Market Microstructure Invariance: Empirical Hypotheses and *Market Microstructure Invariance: A Dynamic Equilibrium Model*

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Invariants in Asset Liquidity

- *Econometrica* paper focuses on comparing asset markets to see how fundamental liquidity determinants affect behavior
- Particular focus: time scale
- Power of small set of determinants to predict across markets constitutes empirical test

Underlying Microstructure

Framework drawn from Treynor 1971.
Two types of trades

- trades due to idiosyncratic needs (effectively, "noise traders")
- speculation based on private information about the underlying value of the asset (effectively, "speculators")

Volume of trades

- Idiosyncratic trades assumed proportional to market capitalization of asset
- Volume of speculation depends on cost of information gathering versus gains from pricing uncertainty versus ability to keep that information from leaking while trading (Grossman Stiglitz, again!)

Underlying Microstructure

- So information gathering by speculators is endogenized; profitability increases activity and causes more rapid dissemination of the information
- Pricing depends on adverse selection, uninformed trader's price will adjust for possibility that counterpart knows more.
- This endogenizes not only mix between informed and uninformed trades, but also size of trades and speed of trading: small trades spread out disguise information (linearity assumption sneaks in here)
- Implications for order size, arrival rate, price impact and bid ask spread.

"Bets"

The fundamental concept: the underlying demand for a trade based on idiosyncratic needs or on gambles on price movements.
Because an investor's bet can be split into multiple trades, spread over time and/or possibly executecd by multiple agents, bets themselves are not observable.

- Let γ_{jt} represent the expected arrival rate (per day, say) of bets on asset j at time t
- Let the random variable $\widetilde{Q}_{jt}$ denote the size of the bet (measured in shares)
  - Assume its expected value is 0
  - Implicitly this is a short term model, so integral of $\widetilde{Q}_{jt}$ over time remains bounded.

Bets and Trade

The "volume" of bets in the market is

$$\gamma_{jt} \cdot E\left\{\left|\widetilde{Q}_{jt}\right|\right\}$$

If individuals split their trades into smaller pieces and traded them twice as quickly, this would not affect the volume of bets.
However trade occuring indirectly through intermediaries, increases volume of trade K&O assume a fixed time-invariant multiplier ζ_j relating the volume of bets to the volume of trade

$$\frac{\zeta_j}{2}\gamma_{jt} \cdot E\left\{\left|\widetilde{Q}_{jt}\right|\right\}$$

- if all betters trade directly with each other then $\zeta_j = 1$.
- if all trades are in the form of a better interacting with an intermediary then $\zeta_j = 2$.
- Free to choose scale so set

$$V_{jt} = \gamma_{jt} \cdot E\left\{\left|\widetilde{Q}_{jt}\right|\right\}$$

Assumptions regarding return volatility

Let σ_{jt} denote the percentage standard deviation of an asset's returns (per day)

- K&O assume constant proportion (!?) of this volatility is due to information leaking through trading (the rest is due to direct release of information–e.g., news)
- Additionally they assume this proportion is time invariant

Consequences:

- Observed price volatility proportional to volatility from bets
- Choose scale so that bet volatility is same as trade volatility.
  - In a unit of time γ_j bets arrive creating $\gamma_j \sigma_{jt}^2$ variance, and $\gamma_j^{1/2} \sigma_{jt}$ standard deviation.
  - K&O prefer to measure in "business time" i.e., per γ_j. Then standard deviation per unit of business time is $\sigma_{jt} / \gamma_j^{1/2}$

Return volatility and risk transfer

Then scaling by the dollar size of the bets, a bet of size $P_{jt} \left| \widetilde{Q}_{jt} \right|$ generates a standard deviation per unit of business time of

$$P_{jt} \left| \widetilde{Q}_{jt} \right| \sigma_{jt} \gamma_j^{-1/2}$$

K&O argue that the signed version of this measures the amount of risk transferred (!?):

$$\widetilde{I}_{jt} = P_{jt} \widetilde{Q}_{jt} \sigma_{jt} \gamma_j^{-1/2}$$

Presumably

- risk transfered is linear in the size of the bet
- risk transfered is linear in the standard deviation associated with bets
- this measure accounts for speed with which bets come to the market: in twice as much time the total standard deviation goes up as the square root of 2.

Invariance of bets

Next K&O hypothesize the "invariance of bets"—that the distibution of $\widetilde{I}$ is constant across time and across assets (a bet in the 99th percentile will transfer risks of the same size in business time). Thus we can drop the subscripts on $\widetilde{I}$

$$\widetilde{I} = P_{jt}\widetilde{Q}_{jt}\sigma_{jt}\gamma_j^{-1/2}$$

Implications:
If scale of stock trading changes, volatility must adjust to make up, per unit of business time.
If the volatility remains the same in calendar time, then the arrival rate of bets must adjust to compensate (as the square).

Testable Implication 1

What we actually see in the market are price P_{jt} volume V_{jt} and volatility σ_{jt}.
The observable equivalent of I is “trading activity” (value adjusted for risk)

$$W_{jt} = \sigma_{jt} P_{jt} V_{jt}$$

Substitution yields that

$$W_{jt} \propto \gamma^{3/2}.$$

Testable implication (not very clearly stated) given three observables, and two equations, triples $\sigma_{jt}, P_{jt}, V_{jt}$ (and the resultant W_{jt}) should always conform to some invariant relation.
Specifically the prediction is that

$$W_j^{2/3}(X_j/V_j)$$

does not vary with stock characteristics, (where X_j is an empirical proxy for $\left|\widetilde{Q}_j\right|$.)

Testable Implication 2

Assume in addition, an invariance in the transactions costs for I, i.e. the dollar cost of a bet depends only on the percentile of the bet in that stock. Then for two different assets, the ratio of transactions costs should be constant at bets scaled to I.
The ratio will depend on a "liquidity measure" specific to each asset. K&O argue that the measure is proportional to

$$[P_{jt} V_{jt} / \sigma^2_{jt}]^{1/3}$$

Intuition for value in brackets: Change a firm from all equity to 50/50 equity riskless debt. No real change, so the $ transactions costs shouldn't change. Size of bets as fraction of value stays the same, but P falls so the number of transactions should double.

Working paper: Agents

A more detailed market microstructure examining these invariances.
Agents:

- Informed traders (get signal of asset value), arrive only once
- Noise traders (for tractability, modeled as having "fake information")
- Risk neutral market makers.

Key feature: informed traders can invest to gather information.
Adverse selection, but costly information gathering. Will compare information gathering decision across assets.
Limitation: informed and noise traders arrive only once in the market.
Informed frequency ought to be endogenous, but noise traders should be exogenous.

Asset pricing

Risk-free rate of interest is 0.
Asset has fundamental price with a geometric Brownian motion $B(t)$

$$F(t) := \exp[\sigma_F \cdot B(t) - \frac{1}{2} \cdot \sigma_F^2 t]$$

So σ_F measures fundamental volatility.
Market makers see price history, and price according to expectation

$$P(t) = E_t\{F(t)| \text{ price history}\}$$

Let

$$\overline{B}(t) = E_t\{B(t)| \text{ price history}\}$$

then the error is

$$B(t) - \overline{B}(t)$$

Asset pricing

Suppose for the moment that the error

$$B(t) - \overline{B}(t)$$

is approximately mean zero with time-varying variance

$$\Sigma(t)/\sigma_F^2.$$

Then

$$P(t) = \exp[\sigma_F \overline{B}(t) + \frac{1}{2}\Sigma(t) - \frac{1}{2} \cdot \sigma_F^2 t]$$

Now conveniently

$$\Sigma(t) = \mathrm{var}_t\{\ln F(t)/P(t)\}$$

so it measures the percentage pricing error.
In short, market makers expectation has an error component whose variance relative to the fundamental variance is measured by $\Sigma(t)$

Role of informed agents

There is a race between changes in fundamentals and adjustments because informed individuals are trading on information, and thereby moving prices towards fundamentals.

- Informed traders arrive at rate $\gamma_I(t)$.
- They pay a fixed amount for information.
- They estimate the fundamental value
- They trade taking account of the effect of their actions on the price.

Specifically, suppose informed trader knows $\Sigma(t)$ and $\overline{B}(t)$. (derivable from price history). In addition, he receives a private signal $\widetilde{i}(t)$.

Role of informed agents

The signal takes the following form:

$$\widetilde{i}(t) = \tau^{1/2}\Sigma(t)^{-1/2}\sigma_F \cdot [B(t) - \overline{B}(t)] + \widetilde{Z}_I(t)$$

Where

- the noise $\widetilde{Z}_I$,is standard normal and independent of history.
- Aside from the noise, the signal estimates how far and in what direction true $B(t)$ differs from market $\overline{B}(t)$.
- τ is precision of the signal (properly speaking it should have a "per unit of time" measurement, but that just means it should be small in a continuous time model; K&O assume it is close to 0).
- the details of the rest of the formula are complex (see the course notes) but they take the form they do simply to ensure that the informed agent's update of his estimate causes a change in $\overline{B}(t)$ which is proportional to $\widetilde{i}(t)$.

Role of informed agents

So the estimate of $\overline{B}(t)$ is updated by the informed trader. Let $\Delta\overline{B}_I(t)$ denote the update:

$$\Delta\overline{B}_I(t) := E\{B(t) - \overline{B}(t)|\widetilde{i}(t)\} = \tau^{1/2}\Sigma(t)^{-1/2}\sigma_F\widetilde{i}(t)$$

This is just an OLS regression argument: we have cov/var as the coefficient on the signal.
Now, using some tricks from Taylor series and continuous approximations, we can describe the effect on the prices if the information $\Delta\overline{B}_I(t)$ were fully incorporated (simply the difference in the expected value with the new information included).

$$E\{F(t) - P(t)|\Delta\overline{B}_I(t)\} \simeq P(t)\sigma_F\Delta\overline{B}_I(t)$$

Role of informed agents

So how much should the informed individual bet? That depends on the effect on prices. His profits, intuitively speaking will be

$$(F - \widehat{P}(Q))Q$$

By $\widehat{P}(.)$ we mean the price he can get the uninformed traders to offer to him for a trade of size Q. Suppose that market makers adjust price in a linear fashion, so that $\widehat{P}(Q) = P - \lambda Q$, where now P is some fixed baseline to be determied. Then

$$E\{(F - P - \lambda Q)Q|\Delta\overline{B}_I(t)\}$$

is the expected profits of the informed trader. Now the informed trader will choose Q to maximize this, so $Q = (F - P)/2\lambda$.

Role of informed agents

Substituting back into the formula for expected profits we have that

$$E\{(F-P-\lambda Q)Q|\Delta \overline{B}_I(t)\} = \frac{(F-P)^2}{4\lambda}$$

What this implies is that activities will adjust so that prices adjust by half of what they would have under full information.
K&O consider a slightly more general possibility of linear adjustment by the fraction θ, making informal arguments about why it might differ from $1/2$.

Volume of trading

Noise traders arrive at a rate such that a constant fraction η of the market turns over per day (per noise trader, so this is okay).
However they also endogenously act so as to make their volumes the same per trade as the informed (this is questionable)
Thus the expected arrival rate of bets is

$$\gamma(t) = \gamma_I(t) + \gamma_U(t)$$

(where I = informed, U = uninformed).
Let $V(t)$ be total volume per day, and let N be the number of shares outstanding. So

$$\gamma_U(t) \cdot E\{\left|\widetilde{Q}(t)\right|\} \text{ and } \gamma(t) E\{\left|\widetilde{Q}(t)\right|\} = V(t)$$

K&O state "the arrival of bets $\gamma(t)$ sets the pace of business time in this model," but it probably ought to be the arrival of informed bets.

Pricing in equilibrium

The market makers are assumed to be using a linear price adjustment rule:

$$\Delta P = \lambda(t)\widetilde{Q}(t)$$

and they set prices equal to the conditional expectation of the fundamental value of the asset.

$$\Delta P = E\{F(t) - P(t)|\widetilde{Q}(t)\}$$

If the market maker knew that a bet was informed, he would undo the price impact for informed traders (adjust by $1/\theta$) and make no adjustment for uninformed traders.

In equilibrium

$$\Delta P = \lambda\widetilde{Q} = \frac{\gamma_I}{\gamma}\lambda\widetilde{Q}\frac{1}{\theta} + \frac{\gamma_U}{\gamma}\lambda\widetilde{Q}\cdot 0$$

In other words, λ cancels out.

The difficulty

The equilibrium does not define λ; instead it restricts the ratio of informed/noise traders.

$$\gamma_I/\gamma = \theta$$

In other words, adverse selection pins down the numbers of informed versus uninformed. K&O provide some speculation about this, but it is not a very satisfactory analysis.

In any event, this means that the volume is

$$V(t) = \eta N/(1-\theta)$$

(Thus the volume ultimately depends on the activity of the noise traders η. Think about what this equation means in terms of θ)

Dynamics

Approximately speaking the amount of impact on price of a trade is

$$\theta P(t)\sigma_F E\{\Delta \overline{B}_I(t)\}$$

and this is true whether the trade is from an informed or an uninformed agent. But the informed make money, while the uninformed lose, and the market makers break even. How does this work? Because of price movements in the future, we can tell if someone was informed or not by the subsequent movements in prices.
But this is a quasi rent for the informed—now we have to include the cost of information gathering—so that profit described before is dissipated in whatever the information cost is.

Dynamics

At this point, K&O flip to a continuous approximation of the model, rather than assuming jumps for each bet. The result of the calculations is that the underlying information (estimates of $\overline{B}$) filters in over time, as does noise. Price follows a martingale incorporating this, but the volatility of price is stochastic. (I think it depends on the pattern of price movements). This means that $\gamma(t)$ itself depends on the volatility at any moment; in some situations it is more valuable for information gathering to occur.

Linking the Two Papers

Finally, they argue that when this model is applied under the case of a constant cost and constant value of the signal, that the model generates the invariance assumptions used in the Econometrica paper. Their intuition for the invariance results is particularly helpful (page 18):

Linking the Two Papers

"Suppose the number of noise traders increases for some exogenous reason. In the structural model this happens when the share price and therefore market capitalization increases, keeping the share turnover of noise traders constant. To be specific,assume that the number of noise traders increases by a factor of 4. As a result, market depth increases and consequently the number of informed traders increases, since their bets now are more profitable. The structural model shows that the number of informed traders eventually increses by a factor of 4 as well, and each of their bets acocunts for a 4 times smaller fraction of returns variance. Returns volatility per unit of business time decreases by a factor of 2 (the square root of 4). The structural model shows that pricing accuracy and liquidity both increase by a factor of 2, as a result of which informed traders exactly cover the cost of private signals by submitting bets 2 times as large as before. Overall dollar volume in the market increases by a factor of 8...One-third of the increase in dollar volume comes from changes in bet size...and two thirds comes from changes in the number of bets..."